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A method for the equivalence group and its infinitesimal generators

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Abstract
We give an effective method for the systematic determination of the equivalence group for any family of differential equations, and for the determination of the infinitesimal generators of this group. This is achieved by viewing the equivalence group as a projected subgroup related to the full symmetry group of the equation, in which the coefficients specifying the family element in the equation are also considered as dependent variables. The method is extended to provide the exact number of fundamental invariants of the equation, without any prior calculation of these invariants. It is then applied to review a number of results obtained in the recent literature on the subject of invariant functions of differential equations. The method is also applied, probably for the first time, to a case of nonlinear equation.

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1. Introduction

One of the most important difficulties that arise in the so-called equivalence problem is the determination of invertible transformations mapping two objects of the same category to each other. Such objects might be for instance definite algebraic structures, differential or algebraic equations or forms, etc. This problem is highly simplified if the two objects can be obtained as members of a restricted and definite family of objects. In this case the relevant transformations must be elements of the equivalence transformation group. Moreover, the invariant functions of the equivalence group can tell *a priori* whether the two objects are not equivalent. These invariants are usually referred to simply as invariants of the corresponding family of objects, and thus we may speak for instance of invariants of differential equations.

In an attempt to generalize some results obtained about the invariants of algebraic forms, invariants of differential equations were introduced in the middle of the 19th century by

Laguerre, Briochi, Halphen and Forsyth, amongst others [1–4]. However, their analysis and corresponding methods were restricted to linear equations, and most often these invariant functions were only semi-invariants. Based on ideas outlined by Lie [5], an infinitesimal method has been suggested, starting with the work of Ovsyannikov [6], and more formally in a much recent paper [7] (see also [3]). However, this method requires the full knowledge of the equivalence transformations (also called *structure invariance group*) of the equation, and it quite often does not yield the right infinitesimal generators. Moreover, the method is quite lengthy, and progress on the subject of these invariant functions has consequently been relatively slow.

All papers dealing with the invariants of differential equations and based on infinitesimal methods are still, as a consequence of the above-mentioned difficulties, restricted to linear equations, and especially classical equations for which the equivalence group is already available [8–12], except perhaps in [13] where an attempt was made to find such a group for a linear equation. The proof to theorem 1 given in the latter paper shows indeed that finding the structure invariance group of a differential equation by a direct application of the definition can be a quite complicated exercise. This direct determination appears to be even more demanding for nonlinear equations. Thus, not only the determination of invariant functions of differential equations has been limited to linear equations alone, but also the results obtained have been rather incomplete, particularly as far as the number of invariants found is concerned.

In this paper, we give a systematic and simple method for finding both the equivalence group and the corresponding infinitesimal generators for any given family of differential equations. This is achieved by viewing the equivalence group of the equation as a projected subgroup associated with the full symmetry group of the equation, in which the arbitrary coefficients specifying the family element are also treated as dependent variables. A simple and precise result concerning the number of fundamental invariants is also given, thus providing a better insight into this determination problem. Examples of application are given by reviewing a number of linear cases treated in the recent literature. A nonlinear case is also considered.

2. The new method of determination

2.1. Generalities

Consider a family $\mathcal{F}$ of differential equations of the form

$$\Delta(x, y_{(n)}; A) = 0, \tag{2.1}$$

where $x = (x_1, \dots, x_p)$ is the set of independent variables, $y = (y_1, \dots, y_q)$ is the set of dependent variables and $y_{(n)}$ denotes the set of all derivatives of y up to the order n, and where $A = (A_1, \dots, A_m)$ is a set of m coefficients specifying the family element in $\mathcal{F}$. Here, the A_j might be either arbitrary functions of x or arbitrary constants. Let G be a connected group of point transformations of the form

$$x = \phi(\bar{x}, \bar{y}; \sigma), \qquad y = \psi(\bar{x}, \bar{y}; \sigma), \tag{2.2}$$

where σ denotes collectively the parameters specifying the group element. These parameters might either be arbitrary functions or arbitrary constants. We say that G is an equivalence group of (2.1) if it maps every element of $\mathcal{F}$ into $\mathcal{F}$. In this case, (2.2) is called the structure invariance group and the transformed equation takes the same form $\Delta(\bar{x}, \bar{y}; \bar{A}) = 0$, where $\bar{A}_j = \bar{A}_j(A, A_{(s)}; \sigma)$ and $A_{(s)}$ represents the set of all derivatives of A up to a certain order s. Consequently, the equivalence group G generates another transformation group G_c acting on the space of coefficients of the original equation [14]. By abuse of language, G_c is often also referred to in the literature as the equivalence group of (2.1). The invariants and semi-invariants of G_c are called the invariants and semi-invariants, respectively, of (2.1). These invariants and semi-invariants are thus functions of the form $\Phi(A, A_{(r)})$ and they satisfy a relation of the form $\Phi(A, A_{(r)}) = \mathbf{w}(\sigma) \cdot \Phi(\bar{A}, \bar{A}_{(r)})$, where the weight $\mathbf{w}$ is exactly 1 when Φ is a true invariant, and not only a semi-invariant. Semi-invariants typically correspond to restricted cases of equivalence groups G that do not involve transformations of either the dependent or the independent variables, and each semi-invariant naturally gives rise to an invariant equation.

In practice, the structure invariance group (2.2) is often given in the more abstract form $x = \phi(\bar{x}, \bar{y}), y = \psi(\bar{x}, \bar{y})$ and does not involve explicitly the group parameter σ. The corresponding infinitesimal transformations can be written in the form

$$\bar{x} \approx x + \varepsilon\xi(x, y), \qquad \bar{y} \approx y + \varepsilon\eta(x, y). \tag{2.3}$$

For convenience, when there is no possibility of confusion, we shall often represent a vector field of the form

$$V = \sum_{i=i}^{s} f^i \partial_{X^i} \qquad \text{by its components :} \quad V = \{f^1, \ldots, f^s\}.$$

Suppose for simplicity that (2.1) is an ODE and let $V^{(n)} = \{\xi, \eta, \zeta_1, \zeta_2, \ldots\}$ be the nth prolongation of $V = \{\xi, \eta\}$. Thus, we have

$$\bar{y}' \approx y' + \varepsilon\zeta_1(x, y, y'), \qquad \bar{y}'' \approx y'' + \varepsilon\zeta_2(x, y, y', y''), \text{ etc.} \tag{2.4}$$

According to the commonly used method (see [3, 10, 8, 11] and the references therein), the original dependent variable y and its derivatives are expressed in terms of $\bar{y}$ and its derivatives according to approximations of the form

$$y' \approx \bar{y}' - \varepsilon\zeta_1(x, y, \bar{y}'), \qquad y'' \approx \bar{y}'' - \varepsilon\zeta_2(x, y, \bar{y}', \bar{y}''), \text{ etc,} \tag{2.5}$$

and these expressions are then used to substitute $\bar{y}$ and its derivatives for y and its derivatives in the original differential equation. This is how the corresponding infinitesimal transformations of the coefficients A of the equation are obtained. However, this method usually yields the wrong generators of the equivalence group, and hence the wrong invariants. For instance, if we let equation (2.1) be a linear first-order equation in three independent variables (x, y, z) and one unknown function u of the form

$$A_1 u_x + A_2 u_y + A_3 u_z = 0, \tag{2.6}$$

and denote by X^0 the corresponding generator of G_c obtained with the usual method, then it was found in [13] that out of the 15 invariants of the second prolongation of X^0, there are 3 of them which are incorrect, namely, the following:

$$\left(\frac{A_2 A_3}{A_1}\right)^2 (A_1)_{y,z}, \qquad \left(\frac{A_3 A_1}{A_2}\right)^2 (A_2)_{x,z}, \qquad \left(\frac{A_1 A_2}{A_3}\right)^2 (A_3)_{x,y},$$

and the correct ones which should replace these three invariants are

$$\left(\frac{A_2 A_3}{A_1}\right) (A_1)_{y,z}, \qquad \left(\frac{A_3 A_1}{A_2}\right) (A_2)_{x,z}, \qquad \left(\frac{A_1 A_2}{A_3}\right) (A_3)_{x,y}.$$

This shows that approximations (2.5) used for transforming the original equation are not appropriate. In fact, in most papers using the infinitesimal methods for finding these invariants, the infinitesimal generators obtained do not match the invariant functions found. The other drawback associated with this method is that not only the full knowledge of the structure invariance group is required, and preferably in infinitesimal form, but also the substitutions of the form (2.5) are recursive and lead to long calculations. It should also be mentioned that even when the structure invariance group is available, the classical point symmetry generator of the equation and its nth prolongation must first be calculated before the method could be applied.

2.2. The proposed method

The method that we are proposing for both the determination of the equivalence group and its generators is based on the very simple observation that the group G of equivalence transformations of (2.1) can be viewed as a certain projection onto the space of the independent and dependent variables $\{x, y\}$ of the full symmetry group G_{S} of (2.1), in which the coefficients $A = (A_1, \dots, A_m)$ are also considered as dependent variables.

Denote by E_σ the group element of G defined by (2.2), and let T_σ be the corresponding element of the group G_c induced by G, and acting on the coefficients A_j of (2.1). Then, with the notation already introduced, T_σ is defined by the transformations $A_j \to \bar{A}_j = \bar{A}_j(A, A_{(s)}; \sigma)$.

Lemma 1. *For every element E_σ of the equivalence group G of (2.1), the set of transformations $S_\sigma = \left\{E_\sigma, T_\sigma^{-1}\right\}$ defines an element of the symmetry group G_{S}.*

Proof. The lemma simply follows from the fact that the group element E_σ keeps the form of the equation invariant, by mapping each element of $\mathcal{F}$ into $\mathcal{F}$, while T_σ acts only on its coefficients. Consequently, once E_σ is applied to the equation, the subsequent application of T_σ^{-1} will bring back the transformed equation to its original form. □

Theorem 1. *Let $X = \{\xi(x, y), \eta(x, y), \phi\}$ be the generic generator of the symmetry group G_{S}, where ϕ denotes collectively the coefficients $\phi_1, \dots, \phi_m$. Let $V^0 = \{\xi^0, \eta^0\}$ be the restriction to $\mathcal{F}$ of the projected generator $V = \{\xi, \eta\}$. That is, V^0 is obtained by the requirement that V generates an element of the equivalence group G. Let ϕ^0 be the resulting restriction on ϕ. That is, $\{\xi^0, \eta^0, \phi^0, \}$ generates the largest symmetry subgroup of G_{S} for which the infinitesimal generator has ξ^0 and η^0 as its first two components. Then the following holds:*

(a) V^0 is the infinitesimal transformation of the structure invariance group (2.2) of G.
(b) $X^0 = \{\xi^0, \phi^0\}$ is the generator of the group G_c.

Proof. The lemma implies that if we impose minimum conditions on V to obtain an infinitesimal generator of G, the resulting generator V^0 must generate the full equivalence group G, and this readily proves part (a) of the theorem. Since the coefficients A_j are supposed to depend only on the independent variables x, and since the transformations of these variables must satisfy the structure invariance group, the generic generator of G_c must be of the stated form X^0. This completes the proof of the theorem. □

In practice, finding the equivalence group of a given family of equations of the form (2.1) reduces according to the theorem to finding the infinitesimal generator $X = \{\xi, \eta, \phi\}$ of the symmetry group of this equation, in which the arbitrary coefficients are treated as dependent variables. Then one looks at minimum conditions that reduce $V = \{\xi, \eta\}$ to a generator $V^0 = \{\xi^0, \eta^0\}$ of the group G of equivalence transformations. These conditions are also imposed on X to obtain ϕ^0. For linear ODEs and PDEs, these minimum conditions can be found by simple inspection and they almost always amount to discarding one of the free constants or functions in the symmetry generator.

The generator $X^0 = \{\xi^0, \phi^0\}$ of the Lie group G_c actually depends linearly on free parameters K_j which are in general functions of x and y and their derivatives, or constants. Moreover, it can always be written as a linear combination of the form

$$X^0 = \sum_{j=1}^{\nu} K_j V_j, \tag{2.7}$$

where the V_j are vector fields which all have the same number of components, and which depend only on x, y, A and the derivatives of the coefficients A. This means, in particular, that the V_j, for $j = 1, \ldots, \nu$, are elements of the Lie algebra L of G_c, because $X^0 \in L$. Moreover, we have the following simple result.

Theorem 2. *Let the generic infinitesimal generator X^0 of an arbitrary Lie group G_c be of the form (2.7).*

(a) A function $F = F(x, y, A)$ is an invariant of G_c if and only if

$$V_j \cdot F = 0, \qquad \text{for all} \quad j = 1, \ldots, \nu. \tag{2.8}$$

(b) The maximal number of fundamental invariants of G_c is $\mathsf{N} - \mathsf{R}$, *where* N *is the number of independent variables in terms of which X^0, and hence each V_j is expressed, while* R *is the rank of the matrix $\mathcal{M}$ whose j th row is V_j.*

(c) When the vector fields V_j, for $j = 1, \ldots, \nu$, generate a Lie subalgebra of the Lie algebra L of G_c, the number of fundamental invariants of G_c is precisely $\mathsf{N} - \mathsf{R}$.

Proof. The condition for F to be an invariant of G_c is $X^0 \cdot F = 0$, and thus part (a) of the theorem readily follows from (2.7) and from the arbitrariness of the parameters K_j. Part (b) is a well-known classical and elementary result concerning the maximal number of functionally independent solutions for systems of PDEs of the form (2.8) (see, for instance [15]). The vector fields V_j do not necessarily form a Lie algebra. However, when they do form a lie algebra as this frequently occurs, the system (2.8) is complete and hence the number of its fundamental invariants is precisely $\mathsf{N} - \mathsf{R}$. □

Although part (a) of the theorem has been consistently applied in most papers in the recent literature on invariant functions [3, 8], the very important but simple result of part (c) of the theorem has not been used in this literature. However, if you know the exact number of invariants beforehand, then you know when you do not need to carry out lengthy calculations to prove that there do not exist nontrivial ones, and you know exactly when you have found them all. The lack of application of parts (b) and (c) of theorem 2 has thus lead to some degree of incompleteness in most papers, and to uselessly long computations (see, for example, the appendix of [11]). Condition (2.8) naturally also holds for differential invariants F of G_c, by substituting the appropriate prolongation for each V_j.

3. Applications

3.1. The second-order linear hyperbolic equation

This partial differential equation, which can be written in terms of the arbitrary coefficients a, b, c in the form

$$u_{x,t} + a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0, \tag{3.1}$$

has been considered in a number of recent papers in connection with the analysis of their invariant functions [7, 8, 10]. In these papers, the focus has been in the determination of a basis of invariants, rather than their number, and this has led to a sequence of different results [8, 10], just perhaps because there is no formal way to ascertain when the right basis has been found. More importantly, the infinitesimal generator of the (full) equivalence group G_c has not been given in most of these papers.

Writing, with the notation already introduced, the generator X of the symmetry group G_S of (3.1) in the form

$$X = \{\xi, \eta, \phi\} \equiv \xi_1\partial_x + \xi_2\partial_t + \eta\partial_u + \phi_1\partial_a + \phi_2\partial_b + \phi_3\partial_c,$$

we find that

$$\xi_1 = f, \tag{3.2a}$$

$$\xi_2 = g, \tag{3.2b}$$

$$\eta = F + uH, \tag{3.2c}$$

$$\phi_1 = -ag_t - H_t, \tag{3.2d}$$

$$\phi_2 = -bf_x - H_x, \tag{3.2e}$$

$$\phi_3 = -\frac{1}{u}(cF + aF_x + bF_t + F_{x,t} + u(c(f_x + g_t) + aH_x + bH_t + H_{x,t})), \tag{3.2f}$$

where $f = f(x)$, $g = g(t)$, $F = F(x,t)$ and $H = H(x,t)$ are all arbitrary functions. The corresponding projection V of X on the space of independent and dependent variables is thus given by $V = \{\xi_1, \xi_2, \eta\}$. It readily follows from equations (3.2*a*)–(3.2*c*) that the only condition to be imposed on V in order to transform it into the infinitesimal generator of the well-known equivalence group of the linear hyperbolic equation (3.1) is to set $F = 0$. Consequently,

$$V^0 = f(x)\partial_x + g(t)\partial_t + uH(x,t)\partial_u$$

and the infinitesimal generator X^0 of G_c takes the form

$$X^0 = f\partial_x + g\partial_t + (-ag_t - H_t)\partial_a + (-bf_x - H_x)\partial_b - (c(f_x + g_t) + aH_x + bH_t + H_{x,t})\partial_c. \tag{3.3}$$

We now set

$$h = a_x + ab - c, \qquad k = b_t + ab - c \quad \text{and} \quad p = k/h.$$

The functions h and k are known as the Laplace invariants, while p is often referred to as the Ovsyannikov invariant [8]. A direct calculation shows that

$$X^0 \cdot p = 0, \qquad X^0 \cdot h = -(f_x + g_t)h, \qquad X^0 \cdot k = -(f_x + g_t)k.$$

This confirms the well-known fact that p is an invariant, while h and k are semi-invariants.

If we now express the first prolongation of X^0 as in (2.7), the corresponding coefficients K_j are made up of the arbitrary functions f, g, H and their derivatives, and there are $\nu = 13$ of them, and the first prolongation of X^0 is expressed in terms of $\mathsf{N} = 11$ independent variables. The matrix $\mathcal{M} = \{V_1, \dots, V_\nu\}$ thus has dimension 13×11, and its rank is $\mathsf{R} = 10$. Therefore, by theorem 2 the exact number of first-order fundamental invariants of (3.1) is one, since V_j generate a Lie algebra in this case. This precise answer completely solves the problem of the first-order differential invariants of (3.1), and gives a better insight into the whole problem. A similar analysis shows that there are no nontrivial zeroth-order invariants.

3.2. *The linear third-order ODE*

Every nth order linear ODE of the form $y^{(n)} = f(x, y, y_{(n-1)})$ can be put into the form

$$y^{(n)} + a_{n-2}y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0, \tag{3.4}$$

where the coefficients a_j for $j = 0, \ldots, n-2$ are arbitrary functions of the independent variable x [4, 2]. Equations written in the form (3.4) are said to be in rational normal form in particular because it does not contain the term $a_{n-1}y^{(n-1)}$, and the structure invariance group for this family of equations is given by

$$x = F(\bar{x}), \qquad y = CF'(\bar{x})^{\frac{n-1}{2}}\bar{y}, \tag{3.5}$$

where F is an arbitrary function and C is an arbitrary parameter.

Pursuing a work undertaken by Laguerre, Briochi, Halphen and others, Forsyth [2] made a brilliant study of (3.4) with regard to their invariant functions and found, in particular, that there are $n-2$ of them, denoted by $\Psi_1, \ldots, \Psi_{n-2}$, which are functionally independent. In the terminology of Forsyth who obtained these functions by a process of differentiation and substitution of various types of semi-invariants that he had found, these invariants were called quotient derivatives [2]. However, his method applies only to linear equations. We shall apply the proposed method here to obtain the sole invariant Ψ_1 in the case of the third-order equation, which is the lowest order linear ODE having nontrivial invariants, and that we may write in the form

$$y^{(3)} + a(x)y' + b(x)y = 0. \tag{3.6}$$

After finding the infinitesimal generator X of the symmetry group G_S of (3.6) as explained above, we see that its projection V on the space on independent and dependent variables x, y has the form

$$V = f(x)\partial_x + [g(x) + y(k_1 + f'(x))]\partial_y,$$

where f and g are arbitrary functions while k_1 is an arbitrary constant. The homogeneity of (3.6) forces g to vanish identically in the expression for V_0, and we thus obtain $V^0 = f\partial_x + y(k_1 + f')\partial_y$, which is clearly the infinitesimal transformation of (3.5) for $n = 3$. The corresponding infinitesimal generator X^0 of the equivalence group G_c takes the form

$$X^0 = f\partial_x - 2(af' + f^{(3)})\partial_a - (3bf' + af'' + F^{(4)})\partial_b. \tag{3.7}$$

Now, we note that in terms of the intermediary function $H(x) = -2b + a'$, an expression for Ψ_1 is given by

$$\Psi_1 = -\frac{4(9aH^2 + 7H'^2 - 6HH'')^3}{H^8}, \tag{3.8}$$

which shows that Ψ_1 is of order (at most) three in the coefficient functions a and b. Writing the third prolongation of X^0 like in (2.7) as a linear combination of the arbitrary function f and its derivatives, we see that the corresponding matrix $\mathcal{M} = \{V_1, \ldots, V_\nu\}$ has dimension 8×9, and rank 8, and the V_j generate a Lie algebra. This, according to theorem 2, shows that (3.6) has only one invariant, as expected. The operators V_1 and V_8 which correspond to the coefficients $K_1 = f$ and $K_8 = f^{(7)}$ simply imply that any invariant is independent of the variables x and $b^{(3)}$. This information helps us to reduce the matrix $\mathcal{M}$ of determining equations for the invariants to the much smaller size 6×7, and we thus obtain

$$\mathcal{M} = \begin{bmatrix} -2a & -3b & -3a_x & -4a_{x,x} & -5a_{x,x,x} & -4b_x & -5b_{x,x} \\ 0 & -a & -2a & -5a_x & -9a_{x,x} & -3b - a_x & -7b_x - a_{x,x} \\ -2 & 0 & 0 & -2a & -7a_x & -a & -3b - 2a_x \\ 0 & -1 & -2 & 0 & -2a & 0 & -a \\ 0 & 0 & 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & -1 \end{bmatrix}. \tag{3.9}$$

Solving the resulting determining equations $V_j \cdot I = 0$, for $j = 1, \ldots, 6$, of the form (2.8), where V_j are determined here by the rows of the matrix $\mathcal{M}$ of (3.9), we obtain the invariant

$$I = \left(\frac{a'}{H}\right)^{2/3} \frac{(9aH^2 + 7H'^2 - 6HH'')}{12H^2(a')^{2/3}}.$$

The functional relation $\Psi_1 = 4 \times (12I)^3$ shows that I is indeed an invariant of (3.6), and that its expression has a much reduced size. Note that the set of independent variables corresponding to the matrix $\mathcal{M}$ of (3.9) is $\{a, b, a', a'', a^{(3)}, b', b''\}$.

3.3. A case of nonlinear equation

Since there is to our knowledge no known case of nonlinear differential equation for which the equivalence group or the corresponding invariants have been determined, we choose for the sake of clarity the elementary family of nonlinear ODEs of the form

$$y'' + a(x)y'^2 = 0, \tag{3.10}$$

where the coefficient $a(x)$ is an arbitrary function. With the same notation as introduced in section 2, we find that the symmetry generator X and the corresponding projection V are given by

$$X = (k_1 + k_2x)\partial_x + f\partial_y - (af' + f'')\partial_a, \qquad V = (k_1 + k_2x)\partial_x + f(y)\partial_y,$$

where k_1 and k_2 are arbitrary constants and $f = f(y)$ is an arbitrary function. Setting $\varphi_t(x, y) = (u(t), W(t))$ for the flow of V in terms of the group parameter t, we see that

$$u = \frac{k_1(-1 + \mathrm{e}^{k_2t}) + k_2\,\mathrm{e}^{k_2t}x}{k_2}, \qquad W = h^{-1}(t + h(y)),$$

where we have set $1/f(y) = h'(y)$. This shows that we may write $x = c_1 + c_2u$, and $y = T(W)$, for some arbitrary constants c_1, c_2 and arbitrary function T. This last substitution with u and W as new independent and new dependent variables, respectively, transforms (3.10) into the form

$$W'^2(aT'^2 + T'') + T'W'' = 0.$$

It therefore follows that $\varphi_t(x, y)$ defines a group of equivalence transformations if and only if $T' = q_2$ is a nonzero constant. Thus, letting q_1 denote another arbitrary constant,

$$x = c_1 + c_2u, \qquad y = q_1 + q_2W, \qquad (c_2 \neq 0, q_2 \neq 0), \tag{3.11}$$

defines the group of equivalence transformations of (3.10). Consequently, in terms of the three arbitrary constants k_1, k_2 and k_3, the generator X^0 of G_c takes the form

$$X^0 = (k_1 + k_2x)\partial_x + ak_3\partial_a.$$

It readily follows that the nonlinear equation (3.10) has neither zeroth-order nor first-order differential invariants, but has a second-order differential invariant

$$I_2 = \frac{a'^2}{aa''}. \tag{3.12}$$

Equations (3.11) and (3.12) constitute probably the first determination of the structure invariance group and corresponding invariants for a nonlinear equation.

All the invariant functions found in this paper were obtained as solutions of the system of linear first-order partial differential equations of the form (2.8). Although there is a growing effort to use alternative methods based on moving frames to find these invariants (see [16–18] and the references therein), the classical method based on the solving of a system PDEs of the form (2.8) is still widely used [3, 7, 8, 10, 11, 19]. Indeed, contrary to the moving frames approach, the essence of the classical method is to reduce the complicated nonlinear equations defining the invariance of a function under a transformation group to the more tractable infinitesimal counterpart given by linear equations of the form (2.8). The algorithm for solving a system of equations of the form (2.8) is naturally much simpler, especially if (2.8) is first reduced to the equivalent adjoint system of total differential equations, in which case the invariants are found essentially by simply solving a sequence of ODEs which are usually simple systems of linear equations. Moreover, the vector fields V_j in (2.8) are not required to generate a Lie algebra, while methods of determination of invariants based on moving frames are designed for the true Lie group or Lie pseudo-group actions, and the normalization equations associated with these methods involve nonlinear equations which are quite often intricate to solve [17]. However, because the moving frames method is intimately associated with Lie group actions, recent research in this direction since the late 1990s has lead to interesting results concerning the properties of submanifolds, the generating systems (or bases) of differential invariants and invariant differential operators associated with transformation groups [17, 18]. In turn, these results have given rise to significant applications in fields related to transformation groups, and such applications include new algorithms for computing symmetry groups and classifying PDEs [20, 21], and the problem of object recognition and symmetry detection based on the characterization of submanifolds via their differential invariant signatures [22] (see [17, 18] for more details regarding these applications).

4. Concluding remarks

We have shown in this paper how to systematically find the equivalence group of a given differential equation and its infinitesimal generators. To give a better insight into the whole determination problem for the invariant functions, we have also shown how the same method can be extended to find the exact number of fundamental invariants of the differential equation, without the need to actually calculate these invariants. The method applies to both linear and nonlinear differential equations, and an example of application to a nonlinear case, perhaps the first of this kind, is considered in the paper.

The method thus proposed does not have applications only in the determination of invariant functions, and its applications in various fields of mathematics and mathematical physics cannot be overstated, simply because of the relevance of the equivalence problem in such fields. However, as far as the problem of determination of the invariants of differential equations is concerned, the efficiency of the new method, which does not require an *a priori* knowledge of the equivalence group contrary to the former infinitesimal method, its accuracy and the simplicity of its implementation, will lead to much faster progress in this area of research. We also note that implementing the method may be made easier in complex cases by the use of specialized computing packages such as MathLie [23], based on the computing system Mathematica, to find the symmetry generator X of G_S.

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